Asymptotic moments of spatial branching processes

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$(\mathtt{P}, \mathtt{G})$-Branching Markov Process

- Particles will live in E a Lusin space (e.g. a Polish space would be enough)
- Let $\mathtt{P} = (\mathtt{P}_t, t \geq 0)$ be a semigroup on E.
- Write $B^+(E)$ for non-negative bounded measurable functions on E

- Particles evolve independently according to a $\mathtt{P}$-Markov process.
- In an event which we refer to as 'branching', particles positioned at x die at rate $\beta \in B^+(E)$ and instantaneously, new particles are created in E according to a point process.
- The configurations of these offspring are described by the random counting measure

$$\mathcal{Z}(A) = \sum_{i=1}^{N} \delta_{x_i}(A),$$

  with probabilities $\mathcal{P}_x$, where $x \in E$ is the position of death of the parent.
- Without loss of generality we can assume that $\mathcal{P}_x(N = 1) = 0$. On the other hand, we do allow for the possibility that $\mathcal{P}_x(N = 0) > 0$ for some or all $x \in E$.
- Henceforth we refer to this spatial branching process as a $(\mathtt{P}, \mathtt{G})$-branching Markov process.

$(\mathtt{P}, \mathtt{G})$-Branching Markov Process

- Define the so-called branching mechanism

$$\mathtt{G}[f](x) := \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N} f(x_i) - f(x)\right], \qquad x \in E,$$

  where we recall $f \in B_1^+(E) := \{f \in B^+(E) : \sup_{x\in E} f(x) \leq 1\}$.
- Configuration of particles at time t is denoted by $\{x_1(t), \ldots, x_{N_t}(t)\}$ and, on the event that the process has not become extinct or exploded,

$$X_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot), \qquad t \geq 0.$$

  is Markovian in $N(E)$, the space of integer atomic measures.
- Its probabilities will be denoted $\mathbb{P} := (\mathbb{P}_\mu, \mu \in N(E))$.
- Define,

$$\mathtt{v}_t[f](x) = \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} f(x_i(t))\right], \qquad f \in B_1^+(E), t \geq 0.$$

- Non-linear evolution semigroup

$$\mathtt{v}_t[f](x) = \mathtt{P}_t[f](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f]]\right](x)\mathrm{d}s, \qquad t \geq 0.$$

k-TH MOMENT

- Our main results concern understanding the growth of the k-th moment functional in time

$$\mathtt{T}_t^{(k)}[f](x) := \mathbb{E}_{\delta_x}[\langle f, X_t\rangle^k], \qquad x \in E, f \in B^+(E), k \geq 1, t \geq 0.$$

- **Notational convenience**: Write $\mathtt{T}_t$ in place of $\mathtt{T}_t^{(1)}$
- Our objective: to show that for $k \geq 2$ and any positive bounded measurable function f on E,

$$\lim_{t\to\infty} g(t)\mathbb{E}_{\delta_x}[\langle f, X_t\rangle^k] = C_k(x, f)$$

  where the constant $C_k(x, f)$ can be identified explicitly.
- We need **two** fundamental assumptions.

Assumption (H1)

There exists an eigenvalue $\lambda \in \mathbb{R}$ and a corresponding right eigenfunction $\varphi \in B^+(E)$ and finite left eigenmeasure $\tilde{\varphi}$ such that, for $f \in B^+(E)$,

$$\langle \mathtt{T}_t[\varphi], \mu \rangle = \mathrm{e}^{\lambda t} \langle \varphi, \mu \rangle \text{ and } \langle \mathtt{T}_t[f], \tilde{\varphi} \rangle = \mathrm{e}^{\lambda t} \langle f, \tilde{\varphi} \rangle,$$

for all $\mu \in N(E)$ if $(X, \mathbb{P})$ is a branching Markov process (resp. a superprocess). Further let us define

$$\Delta_t = \sup_{x \in E, f \in B_1^+(E)} |\varphi(x)^{-1} \mathrm{e}^{-\lambda t} \mathtt{T}_t[f](x) - \langle \tilde{\varphi}, f \rangle|, \qquad t \geq 0.$$

We suppose that

$$\sup_{t \geq 0} \Delta_t < \infty \text{ and } \lim_{t \to \infty} \Delta_t = 0.$$

NOTE: This assumption allows us to talk about criticality of the $(\mathtt{P}, \mathtt{G})$-BMP:

$$\lambda = 0 \text{ (critical)} \mid \lambda > 0 \text{ (supercritical)} \mid \lambda < 0 \text{ (subcritical)}$$

ASSUMPTION (H2)$_k$

$$\sup_{x \in E} \mathcal{E}_x(\langle 1, \mathcal{Z} \rangle^k) < \infty.$$

THEOREM: THE CRITICAL CASE ($\lambda = 0$)

Suppose that (H1) holds along with $(\mathrm{H2})_k$ for some $k \geq 2$ and $\lambda = 0$. Define

$$\Delta_t^{(\ell)} = \sup_{x \in E, f \in B_1^+(E)} \left| t^{-(\ell-1)} \varphi(x)^{-1} \mathtt{T}_t^{(\ell)}[f](x) - 2^{-(\ell-1)} \ell! \, \langle f, \tilde{\varphi} \rangle^\ell \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle^{\ell-1} \right|,$$

where

$$\mathbb{V}[\varphi](x) = \beta(x) \mathcal{E}_x \left(\langle \varphi, \mathcal{Z} \rangle^2 - \langle \varphi^2, \mathcal{Z} \rangle \right).$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

In short, subject to (H1) at criticality and $(\mathrm{H2})_k$, we have, for $f \in B_1^+(E)$,

$$\lim_{t \to \infty} t^{-(k-1)} \mathbb{E}_{\delta_x} \left[\langle f, X_t \rangle^k \right] = 2^{-(k-1)} \ell! \, \langle f, \tilde{\varphi} \rangle^k \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle^{k-1} \varphi(x)$$

"At criticality the k-th moment scales like t^{k-1}"

Ideas from the proof

- The obvious starting point:

$$\mathtt{T}^{(k)}[f](x) = (-1)^k \frac{\partial}{\partial\theta}\mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta\langle f, X_t\rangle}]\Big|_{\theta=0}$$

- Recall that

$$\mathtt{v}_t[f](x) = \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} f(x_i(t))\right], \qquad f \in B_1^+(E), t \geq 0.$$

- Non-linear evolution semigroup

$$\mathtt{v}_t[f](x) = \mathtt{P}_t[f](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f]]\right](x)\mathrm{d}s, \qquad t \geq 0.$$

- Hence

$$\mathtt{v}_t[\mathrm{e}^{-\theta f}](x) = \mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta\langle f, X_t\rangle}]$$

- We need a new representation of the non-linear semigroup $(\mathtt{v}_t, t \geq 0)$ which connects us to the assumption (H1).

Linear to non-linear semigroup

- Recall

$$\mathtt{T}_t[f](x) = \mathtt{T}_t^{(1)}[f](x) = \mathbb{E}_{\delta_x}[\langle f, X_t\rangle], \qquad t \geq 0, f \in B_1^+(E), x \in E.$$

- For $f \in B^+(E)$, it is well known that the mean semigroup evolution satisfies

$$\mathtt{T}_t[f](x) = \mathtt{P}_t[f] + \int_0^t \mathtt{P}_s\left[\mathtt{F}\,\mathtt{T}_{t-s}[f]\right](x)\mathrm{d}s \qquad t \geq 0, x \in E, \tag{1}$$

where

$$\mathtt{F}[f](x) = \beta(x)\mathcal{E}_x\left[\sum_{i=1}^{N} f(x_i) - f(x)\right] =: \beta(x)(\mathtt{m}[f](x) - f(x)), \qquad x \in E.$$

Linear to non-linear semigroup

We now define a variant of the non-linear evolution semigroup equation

$$\mathtt{u}_t[f](x) = \mathbb{E}_{\delta_x}\left[1 - \prod_{i=1}^{N_t} f(x_i(t))\right], \qquad t \geq 0,\ x \in E,\ f \in B_1^+(E).$$

For $f \in B_1^+(E)$, define

$$\mathtt{A}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N}(1 - f(x_i)) - 1 + \sum_{i=1}^{N} f(x_i)\right], \qquad x \in E.$$

$$\mathtt{v}_t[f](x) = \mathtt{P}_t[f](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f]]\right](x)\mathrm{d}s \quad \text{and} \quad \mathtt{T}_t[f](x) = \mathtt{P}_t[f] + \int_0^t \mathtt{P}_s\left[\mathtt{F}\mathtt{T}_{t-s}[f]\right](x)\mathrm{d}s$$

gives us.....

Lemma

For all $g \in B_1^+(E)$, $x \in E$ and $t \geq 0$, the non-linear semigroup $\mathtt{u}_t[g](x)$ satisfies

$$\mathtt{u}_t[g](x) = \mathtt{T}_t[1 - g](x) - \int_0^t \mathtt{T}_s\left[\mathtt{A}[\mathtt{u}_{t-s}[g]]\right](x)\mathrm{d}s.$$

Nonlinear to k-th moment evolution equation

In terms of our new semigroup equation:

$$\mathtt{T}_t^{(k)}[f](x) = (-1)^{k+1} \frac{\partial^k}{\partial \theta^k} \mathtt{u}_t[\mathrm{e}^{-\theta f}](x)\bigg|_{\theta=0}.$$

Theorem

Fix $k \geq 2$. Assuming (H1) and (H2)$_k$, with the additional assumption that

$$\sup_{x\in E, s\leq t} \mathtt{T}_s^{(\ell)}[f](x) < \infty, \qquad \ell \leq k-1, f \in B^+(E), t \geq 0, \tag{2}$$

it holds that

$$\mathtt{T}_t^{(k)}[f](x) = \mathtt{T}_t[f^k](x) + \int_0^t \mathtt{T}_s\left[\beta \eta_{t-s}^{(k-1)}[f]\right](x)\,\mathrm{d}s, \qquad t \geq 0, \tag{3}$$

where

$$\eta_{t-s}^{(k-1)}[f](x) = \mathcal{E}_x\left[\sum_{[k_1,\ldots,k_N]_k^2} \binom{k}{k_1,\ldots,k_N} \prod_{j=1}^N \mathtt{T}_{t-s}^{(k_j)}[f](x_j)\right],$$

and $[k_1,\ldots,k_N]_k^2$ is the set of all non-negative N-tuples $(k_1,\ldots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the k_i are strictly positive.

Induction: $k \mapsto k+1$

- Suppose the result is true for the first k moments.
- Recall $\mathtt{T}_t[f](x) \to \langle f, \tilde{\varphi}\rangle \varphi(x)$ so that, for $k \geq 2$,

$$\lim_{t\to\infty} t^{-k}\mathtt{T}_t[f^{k+1}](x) \to 0$$

- Hence:

$$\lim_{t\to\infty} t^{-k}\mathtt{T}_t^{(k+1)}[f](x)$$

$$= \lim_{t\to\infty} t^{-k}\int_0^t \mathtt{T}_s\left[\mathcal{E}.\left[\sum_{[k_1,\ldots,k_N]^2_{k+1}} \binom{k+1}{k_1,\ldots,k_N}\prod_{j=1}^N \mathtt{T}_{t-s}^{(k_j)}[f](x_j)\right]\right](x)\mathrm{d}s$$

$$= \lim_{t\to\infty} t^{-(k-1)}\int_0^1 \mathtt{T}_{ut}\left[\mathcal{E}.\left[\sum_{[k_1,\ldots,k_N]^2_{k+1}} \binom{k+1}{k_1,\ldots,k_N}\prod_{j=1}^N \mathtt{T}_{t(1-u)}^{(k_j)}[f](x_j)\right]\right](x)\mathrm{d}u$$

$$= \lim_{t\to\infty} \int_0^1 \mathtt{T}_{ut}\left[\mathcal{E}.\left[\sum_{[k_1,\ldots,k_N]^2_{k+1}} \binom{k+1}{k_1,\ldots,k_N}\frac{(t(1-u))^{k+1-\#\{j:k_j>0\}}}{t^{k-1}}\prod_{j=1}^N \frac{\mathtt{T}_{t(1-u)}^{(k_j)}[f](x_j)}{(t(1-u))^{k_j-1}}\right]\right](x)\mathrm{d}u$$

Rough outline of the induction: $k \mapsto k+1$

- From the last slide:

$$\lim_{t\to\infty} t^{-k}\mathtt{T}_t^{(k+1)}[f](x)$$

$$= \lim_{t\to\infty}\int_0^1 \mathtt{T}_{ut}\left[\mathcal{E}.\left[\sum_{[k_1,\dots,k_N]^2_{k+1}} \binom{k+1}{k_1,\dots,k_N}\frac{(t(1-u))^{k+1-\#\{j:k_j>0\}}}{t^{k-1}}\prod_{j=1}^N \frac{\mathtt{T}^{(k_j)}_{t(1-u)}[f](x_j)}{(t(1-u))^{k_j-1}}\right]\right](x)\mathrm{d}u$$

- Largest terms in blue correspond to those summands for which $\#\{j : k_j > 0\} = 2$
- The induction hypothesis plus $\sum_{i=1}^N k_j = k+1$ ensures that the product term is asymptotically a constant
- The simple identity

$$\sum_{[k_1,\dots,k_N]^2_{k+1}} \binom{k+1}{k_1,\dots,k_N} \le N^{k+1}$$

  shows us where the need for the hypothesis (H2) comes in.
- We need an ergodic limit theorem that reads (roughly): If

$$F(x,u) := \lim_{t\to\infty} F(x,u,t), \qquad x\in E, u\in[0,1],$$

  "uniformly" for $(u,x)\in[0,1]\times E$, then

$$\lim_{t\to\infty}\int_0^1 \mathtt{T}_{ut}[F(\cdot,u,t)](x)\mathrm{d}u = \int_0^1 \langle\tilde{\varphi}, F(\cdot,u)\rangle \mathrm{d}u$$

  "uniformly" for $x\in E$.

THEOREM: SUPERCRITICAL ($\lambda > 0$)

Suppose that (H1) holds along with $(\mathrm{H2})_k$ for some $k \geq 2$ and $\lambda > 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, f \in B_1^+(E)} \left| \varphi(x)^{-1} \mathrm{e}^{-\ell\lambda t} \mathtt{T}_t^{(\ell)}[f](x) - \ell! \langle f, \tilde{\varphi} \rangle^\ell L_\ell \right|,$$

where $L_1(x) = 1$ and we define iteratively for $k \geq 2$,

$$L_k = \frac{1}{\lambda(k-1)} \left\langle \tilde{\varphi}, \beta \mathcal{E}. \left[\sum_{[k_1, \dots, k_N]_k^2} \prod_{\substack{j=1 \\ j : k_j > 0}}^{N} \varphi(x_j) L_{k_j} \right] \right\rangle.$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

"At subcriticality the k-th moment scales like $\mathrm{e}^{\lambda k t}$ (i.e. the first moment to the power k)"

Theorem (Subcritical, $\lambda < 0$)

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, f \in B_1^+(E)} \left| \varphi(x)^{-1} \mathrm{e}^{-\lambda t} \mathrm{T}_t^{(\ell)}[f](x) - \ell! \langle f, \tilde{\varphi} \rangle^{\ell} L_\ell \right|,$$

where we define iteratively $L_1 = \langle f, \tilde{\varphi} \rangle$ and for $k \geq 2$,

$$L_k = \frac{\langle f^k, \tilde{\varphi} \rangle}{\langle f, \tilde{\varphi} \rangle^k k!} - \left\langle \beta \mathcal{E}. \left[\sum_{n=2}^{k} \frac{1}{\lambda(n-1)} \sum_{[k_1, \ldots, k_N]_k^n} \prod_{\substack{j=1 \\ j : k_j > 0}}^{N} \varphi(x_j) L_{k_j} \right], \tilde{\varphi} \right\rangle.$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

"At subcriticality the k-th moment scales like $\mathrm{e}^{\lambda t}$ (i.e. like the first moment)"

WHAT ABOUT SUPERPROCESSES?

- A Markov process $X := (X_t : t \geq 0)$ on $M(E)$, the space of finite measures on Lusin space E, with $\mathbb{P} := (\mathbb{P}_\mu, \mu \in M(E))$.
- Transition semigroup

$$\mathbb{E}_\mu \left[e^{-\langle f, X_t \rangle} \right] = e^{-\langle \mathtt{V}_t[f], \mu \rangle}, \qquad \mu \in M(E), f \in B^+(E),$$

where

$$\mathtt{V}_t[f](x) = \mathtt{P}_t[f](x) - \int_0^t \mathtt{P}_s \left\langle \psi(\cdot, \mathtt{V}_{t-s}[f](\cdot)) + \phi(\cdot, \mathtt{V}_{t-s}[f]) \right| (x) \mathrm{d}s.$$

- Here ψ denotes the local branching mechanism

$$\psi(x, \lambda) = -b(x)\lambda + c(x)\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda y} - 1 + \lambda y \right) \nu(x, \mathrm{d}y), \quad \lambda \geq 0, \tag{4}$$

where $b \in B(E)$, $c \in B^+(E)$ and $(x \wedge x^2)\nu(x, \mathrm{d}y)$ is a bounded kernel from E to $(0, \infty)$, and ϕ is the non-local branching mechanism

$$\phi(x, f) = \beta(x)f(x) - \beta(x)\gamma(x, f) - \beta(x) \int_{M(E)^\circ} (1 - e^{-\langle f, \nu \rangle}) \Gamma(x, \mathrm{d}\nu),$$

where $\beta \in B^+(E)$, $\gamma(x, f)$ is a bounded function on $E \times B^+(E)$ and $\nu(1)\Gamma(x, \mathrm{d}\nu)$ is a bounded kernel from E to $M(E)^\circ := M(E) \setminus \{ 0 \}$ with

$$\gamma(x, f) + \int_{M(E)^\circ} \langle 1, \nu \rangle \Gamma(x, \mathrm{d}\nu) \leq 1.$$

What about superprocesses?

• Keep the same notation e.g.

$$\mathtt{T}_t^{(k)}[f](x) := \mathbb{E}_{\delta_x}[\langle f, X_t\rangle^k], \qquad x \in E, f \in B^+(E), k \geq 1, t \geq 0.$$

• Under the same first ergodic moment assumption (H1) and (H2)$_k$ replaced by

$$\sup_{x\in E}\left(\int_0^\infty |y|^k \nu(x, \mathrm{d}y) + \int_{M(E)^\circ} \langle 1, \nu\rangle^k \Gamma(x, \mathrm{d}\nu)\right) < \infty.$$

• A different proof is needed because we cannot work under the expectation with individual particles.

• Instead an approach using Faa di Bruno's formula can be used taking advantage of the smoother branching mechanism than in the particle setting.

• The same conclusions hold for the critical, supercritical and subcritical setting as for the branching particle setting, albeit the constants in the limit are slightly different.

Thank you!